Dynamics of test bodies in scalar-tensor theory and equivalence principle

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Abstract

How do test bodies move in scalar-tensor theories of gravitation? We provide an answer to this question on the basis of a unified multipolar scheme. In particular, we give the explicit equations of motion for point-like, as well as spinning test bodies, thus extending the well-known general relativistic results of Mathisson, Papapetrou, and Dixon to scalar-tensor theories of gravity. We demonstrate the validity of the equivalence principle for test bodies.

1 Introduction

Scalar-tensor theories are considered to be close and viable generalizations of Einstein's general relativity theory. Since their introduction in [17, 18, 26] they attracted a lot of attention after the works of Brans and Dicke [5, 2, 3, 7, 8] with the scalar field interpreted as a variable gravitational coupling – for an overview of the history and results of scalar-tensor theories see [11, 4, 12, 24].

However, little attention was paid to the study of motion of extended test bodies in such theories. Following [3, 1, 27], the issue was thoroughly analyzed [6] in the framework of the post-Newtonian formalism. Here we present the test body dynamics for a very large class of scalar-tensor theories.

2 Scalar-tensor theory

We study a class of scalar-tensor theories (along the lines of [6]) with the action $I = \int d^4x \overset{J}{\mathfrak{L}}$ on the spacetime manifold with the metric $\overset{J}{g}_{ij}$. The Lagrangian density $\overset{J}{\mathfrak{L}} = \sqrt{-\overset{J}{g}}\overset{J}{L}$ has the following form

$$\overset{J}{L} = \frac{1}{2\kappa}\left(-F^2 R(\overset{J}{g}) + \overset{J}{g}^{ij}\overset{J}{\gamma}_{AB}\partial_i\varphi^A\partial_j\varphi^B - 2\overset{J}{U}\right) + L_{\mathrm{m}}(\psi, \partial\psi, \overset{J}{g}_{ij}). \tag{1}$$

This extends the Brans-Dicke theory [28, 29] to the case with a multiplet of scalar fields φ^A (capital indices $A, B, C = 1, \ldots, N$ label the components of the multiplet). Here $\kappa = 8\pi G/c^4$ is Einstein's gravitational constant and

$$F = F(\varphi^A), \qquad \overset{J}{U} = \overset{J}{U}(\varphi^A), \qquad \overset{J}{\gamma}_{AB} = \overset{J}{\gamma}_{AB}(\varphi^A). \tag{2}$$

The Lagrangian $L_{\mathrm{m}}(\psi, \partial\psi, \overset{J}{g}_{ij})$ depends on the matter fields ψ.

The metric $\overset{J}{g}_{ij}$ determines angles and intervals in the *Jordan reference frame.* The Riemannian curvature scalar $R(\overset{J}{g})$ is constructed from the Jordan metric. With the help of the conformal transformation

$$\overset{J}{g}_{ij} \longrightarrow g_{ij} = F^2 \overset{J}{g}_{ij} \tag{3}$$

we obtain the metric in the *Einstein reference frame.*

In the Einstein reference frame the Lagrangian density in the scalar-tensor theory reads $\mathfrak{L} = \sqrt{-g}L$ with

$$L = \frac{1}{2\kappa}\left(-R + g^{ij}\gamma_{AB}\partial_i\varphi^A\partial_j\varphi^B - 2U\right) + \frac{1}{F^4}L_{\mathrm{mat}}(\psi, \partial\psi, F^{-2}g_{ij}). \tag{4}$$

The scalar curvature $R(g)$ is constructed from the Einstein metric g_{ij}, and

$$\gamma_{AB} = \frac{1}{F^2}(\overset{J}{\gamma}_{AB} + 6F_{,A}F_{,B}), \qquad U = \frac{1}{F^4}\overset{J}{U}. \tag{5}$$

The metrical energy-momentum tensor of matter is defined by $\sqrt{-g}t_{ij} := 2\partial(\sqrt{-g}L_{\mathrm{mat}})/\partial g^{ij}$. The Noether theorem yields the conservation law

$$\nabla_j t^{kj} = \frac{1}{F}\left(4t^{kj} - g^{kj}g_{mn}t^{mn}\right)\partial_j F = -V_{ij}{}^k t^{ij}. \tag{6}$$

For details see [20]. Here $V_{ij}{}^k = A_j\delta_i^k - \frac{1}{4}g_{ij}A^k$, and $A_i := \partial_i \log F^{-4}$.

3 Equations of motion

We derive the equations of motion in the Mathisson-Papapetrou-Dixon [19, 21, 9, 10] approach by integrating the conservation law (6) using the geodesic

expansion technique of Synge [25]. With the world function σ and the parallel propagator by $g^y{}_x$, we introduce integrated moments to an arbitrary order $n = 0, 1, 2, \ldots$ by:

$$p^{y_1 \cdots y_n y_0} \quad := \quad (-1)^n \int_{\Sigma(s)} \sigma^{y_1} \cdots \sigma^{y_n} g^{y_0}{}_{x_0} \sqrt{-g} t^{x_0 x_1} d\Sigma_{x_1}, \tag{7}$$

$$k^{y_2 \cdots y_{n+1} y_0 y_1} \quad := \quad (-1)^n \int_{\Sigma(s)} \sigma^{y_1} \cdots \sigma^{y_n} g^{y_0}{}_{x_0} g^{y_1}{}_{x_1} \sqrt{-g} t^{x_0 x_1} w^{x_2} d\Sigma_{x_2}. \tag{8}$$

We use a condensed notation so that y_n denotes indices at the point y which we associate with the world-line $y(s)$ of a test body, parametrized by the proper time s. The integrals are performed over spatial hypersurfaces $\Sigma(s)$.

4 Pole-dipole equations of motion

In the pole-dipole approximation, an extended body is characterized by the multipole moments $p^a, p^{ab}, k^{ab}, k^{abc}$. Using the general multipolar scheme [23] we derive the equations of motion for these moments:

$$0 \quad = \quad k^{(a|c|b)} - v^{(a} p^{b)c}, \tag{9}$$

$$\frac{D}{ds} p^{ab} \quad = \quad k^{ba} - v^a p^b - V_{dc}{}^b k^{acd}, \tag{10}$$

$$\frac{D}{ds} p^a \quad = \quad -V_{cb}{}^a k^{bc} - V_{dc}{}^a{}_{;b} k^{bcd} - \frac{1}{2} R^a{}_{cdb} \left(k^{bcd} + v^d p^{bc} \right). \tag{11}$$

Here $v^a := dy^a/ds$ denotes the normalized four-velocity of a body. Since $k^{a[bc]} = 0$, we can solve (9) to find explicitly

$$k^{abc} = v^a p^{cb} + v^c p^{[ab]} + v^b p^{[ac]} + v^a p^{[bc]}. \tag{12}$$

Plugging this into (10) and (11), we obtain the generalized Mathisson-Papapetrou-Dixon system

$$\frac{D\mathcal{P}^a}{ds} \quad = \quad \frac{1}{2} R^a{}_{bcd} v^b \mathcal{J}^{cd} - \xi \nabla^a F^{-4} - \xi^b \nabla_b \nabla^a F^{-4}, \tag{13}$$

$$\frac{D\mathcal{J}^{ab}}{ds} \quad = \quad -2 v^{[a} \mathcal{P}^{b]} - 2 \xi^{[a} \nabla^{b]} F^{-4}. \tag{14}$$

Here, following [22, 23, 20], we introduce the generalized total energy–momentum 4-vector and the generalized total angular momentum by

$$\mathcal{P}^a \quad := \quad F^{-4} p^a + p^{ba} \nabla_b F^{-4}, \tag{15}$$

$$\mathcal{J}^{ab} \quad := \quad F^{-4} L^{ab}. \tag{16}$$

The orbital angular moment is defined by $L^{ab} := 2 p^{[ab]}$, and we denoted

$$\xi^a = -\frac{1}{4} g_{bc} k^{abc}, \qquad \xi = -\frac{1}{4} g_{ab} k^{ab}. \tag{17}$$

5 Monopolar equations of motion

At the monopolar order, the only nontrivial moments are p^a, and k^{ab}. The system (9)-(11) then reduces to

$$0 = k^{ba} - v^a p^b, \tag{18}$$

$$\frac{Dp^a}{ds} = -V_{cb}{}^a k^{bc}. \tag{19}$$

Making use of $k^{[ab]} = 0$, the first equation yields $v^{[a} p^{b]} = 0$, hence we have

$$p^a = M v^a \tag{20}$$

with the mass $M := v^a p_a$. Substituting (18) and (20) into (19) we find $\xi = -\frac{v^a p_a}{4} = -\frac{M}{4}$ and

$$\frac{Dv^a}{ds} = -(g^{ab} - v^a v^b)\frac{\nabla_b F}{F}. \tag{21}$$

Quite remarkably, the dynamics of an extended test body in the monopole approximation is independent of body's mass. For a trivial coupling function F, equation (21) reproduces the general relativistic result. Interestingly, the mass of a body is not constant: $M = F^3 M_0$ with $M_0 = \text{const}$.

6 Conclusions

Our main result is the system (13)-(14) that describes the dynamics of extended test bodies in scalar-tensor gravity. In the monopolar case, our analysis revealed a surprisingly simple equation of motion (21). In contrast to geodesic motion in General Relativity, freely falling massive test bodies in scalar-tensor gravity experience an additional force, determined by the new scalar degrees of freedom encoded in the function F.

A remarkable feature of (21) is the prediction that all massive test bodies move in the same way, independently of their mass. We thus demonstrate the validity of the equivalence principle in scalar-tensor gravity. This is consistent with the previous observation [15, 16] that the total scalar charge of a body is equal to its mass when the scalar field self-interactions are neglected. The latter is in agreement with the test body assumption that underlies the Mathisson-Papapetrou-Dixon approach. When one goes beyond the test body approximation, however, the scalar charge is no longer equal to the mass and a further study is needed to fix their relation [13, 14].

References

[1] P. Bergmann, *Int. J. Theor. Phys.*, **1**:25, 1968.

[2] C. Brans, *Phys. Rev.*, **125**:2194, 1962.

[3] C. Brans, *Phys. Rev.*, **125**:388, 1962.

[4] C. Brans, *arXiv:gr-qc/0506063*, 2005.

[5] C. Brans and R. H. Dicke, *Phys. Rev.*, **124**:925, 1961.

[6] T. Damour and G. Esposito-Farèse, *Class. Quantum Grav.*, **9**:2093, 1992.

[7] R. H. Dicke, *Phys. Rev.*, **125**:2163, 1962.

[8] R. H. Dicke, *The theoretical significance of experimental relativity* (Gordon and Breach: New York, 1964).

[9] W. G. Dixon, *Nuovo Cimento*, **34**:317, 1964.

[10] W. G. Dixon, *Phil. Trans. R. Soc. Lon. A*, **277**:59, 1974.

[11] Y. Fujii and K. Maeda, *The scalar-tensor theory of gravity* (Cambridge University Press: Cambridge, 2003).

[12] H. Goenner, *Gen. Rel. Grav.*, **44**:2077, 2012.

[13] S. E. Gralla, *Phys. Rev. D*, **81**:084060, 2010.

[14] S. E. Gralla, *Phys. Rev. D*, **87**:104020, 2013.

[15] L. Hui and A. Nicolis, *Phys. Rev. Lett.*, **105**:231101, 2010.

[16] L. Hui, A. Nicolis, and C. W. Stubbs, *Phys. Rev. D*, **80**:104002, 2009.

[17] P. Jordan, *Schwerkraft und Weltall* (Vieweg: Braunschweig, 1955).

[18] P. Jordan, *Z. Phys.*, **157**:112, 1959.

[19] M. Mathisson, *Acta Phys. Pol.*, **6**:163, 1937.

[20] Y. N. Obukhov and D. Puetzfeld, *Phys. Rev. D*, **90**:104041, 2014.

[21] A. Papapetrou, *Proc. Roy. Soc. Lond. A*, **209**:248, 1951.

[22] D. Puetzfeld and Y. N. Obukhov, *Phys. Rev. D*, **88**:064025, 2013.

[23] D. Puetzfeld and Y. N. Obukhov, *Phys. Rev. D*, **90**:084034, 2014.

[24] T. P. Sotiriou, *Lect. Notes Phys.*, Ed. E. Papantonopoulos, **892**:3, 2015.

[25] J. L. Synge, *Relativity: The general theory* (North-Holland: Amsterdam, 1960).

[26] Y. Thiry, *J. Math. Pures et Appl., sér. 9*, **30**:275, 1951.

[27] R. V. Wagoner, *Phys. Rev.*, **12**:3209, 1970.

[28] C. M. Will, *Theory and experiment in gravitational physics* (Cambridge University Press: Cambridge, 1993).

[29] C. M. Will, *Living Reviews in Relativity*, 17(4), 2014.